Weak insensitivity to initial conditions at the edge of chaos in the logistic map [1]

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Abstract

We extend existing studies of weakly sensitive points within the framework of Tsallis non-extensive thermodynamics to include weakly insensitive points at the edge of chaos. Analyzing tangent points of the logistic map we have verified that the generalized entropy with suitable entropic index q correctly describes the approach to the attractor.

1 Introduction

Strong sensitivity to initial conditions is one of the main features of chaos. In the simple case of a one-dimensional dynamical variable x the sensitivity function is $\xi(t) = \Delta x(t)/\Delta x(0) \sim e^{\lambda t}$ for $\Delta x(0) \to 0$ and $t \to \infty$, where a positive (negative) Liapunov exponent λ characterizes strong sensitivity (insensitivity) to initial conditions. The marginal case $\lambda = 0$ has already been treated in the framework of non-extensive thermodynamics [1,2,3,4,5], introduced by Tsallis [6] to describe systems with long-range interactions or fractal space-time structures, see Ref. [7] and references therein. A suitable generalization of the sensitivity function is [1,2]:

$$\xi(t) = \lim_{\Delta x(0)\to 0} \lim_{t\to\infty} \frac{\Delta x(t)}{\Delta x(0)} \sim [1 + (1 - q_{sen})\lambda_{q_{sen}} t]^{1/(1-q_{sen})} \quad . \tag{1}$$

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The parameter q_{sen} is related to the entropic index q, that controls the degree of non-extensivity in the entropy introduced by Tsallis ($k_B = 1$):

$$S_q = \frac{1 - \sum p_i^q}{q - 1} \,. \tag{2}$$

The usual exponential sensitivity, $\xi \sim e^{\lambda t}$, and the standard extensive entropy, $S_1 = -\sum p_i \ln p_i$, are reproduced in the $q \to 1$ limit, while an entropic index $q > 1$ ($q < 1$) characterizes the so-called weak insensitivity (sensitivity) to initial conditions [1,2]. The logistic map

$$x_{t+1} = f(x_t) = 1 - \mu x_t^2 \quad ; \quad -1 \le x_t \le 1 \,,\; 0 \le \mu \le 2 \,,\; t = 0, 1, 2, \ldots \tag{3}$$

is a simple one-dimensional system that can be used to investigate the connection between dynamical behavior and non-extensivity at the edge of chaos. The Liapunov exponent vanishes for specific values of the parameter μ: these points can be 2^∞ bifurcation critical points, which are weakly sensitive ($q < 1$), and periodic and tangent bifurcation points, which are weakly insensitive ($q > 1$). Transition from periodic to chaotic behavior at the 2^∞ bifurcation critical points (μ_c) involves a Feigenbaum-scaling cascade of period doubling, while the opposite transition at the tangent bifurcation points (μ_t) shows intermittency. In spite of the great amount of work on the weakly-sensitive critical points, μ_c, leading to the determination of the entropic index using different approaches [2,3,4,8,10,11], much less attention has been given to weakly-insensitive points at the edge of chaos: these latter points are the subject of our study.

2 Numerical analysis

For μ that satisfies $f_\mu^{(k)}(x) = x$ and $|f_\mu^{(k)'}(x)| = 1$, the Liapunov exponent $\lambda = 0$ and $\xi(t)$ of Eq. (1) has a power-law behavior with $q_{sen} > 1$ (weak insensitivity). All the bifurcation points and the tangent points μ_t at the beginning of the periodic windows (at the edge of chaos) belong to this ensemble. The value of q_{sen} can be derived within the continuous limit approximation [1,5]; for instance, $q_{sen} = 3/2$, or $(1 - q_{sen})^{-1} = -2$, at the beginning of the period-three window ($\mu_t = 7/4$), while $q_{sen} = 5/3$, or $(1 - q_{sen})^{-1} = -3/2$, at the first bifurcation point of the main sequence ($\mu = 3/4$).

Calculating the logarithm of $\xi(t)$, approximated as $\sum_{i=1}^{N-1} \ln(|2\mu x_i|)$, for the first $N \gg 1$ time steps, we verified that the sensitivity function decreases as a power. In Fig. 1 $\ln(\xi(N))$ is plotted versus $\ln(N)$ for an ensemble of starting points x_0. The system shows two regimes: (1) at the beginning it converges to

the attractor (the three almost-stable solutions of the equation $f^{(3)}_{\mu=7/4}(x) = x$); (2) then $\xi(N)$ shows the expected asymptotic behavior $\sim N^{-2}$.

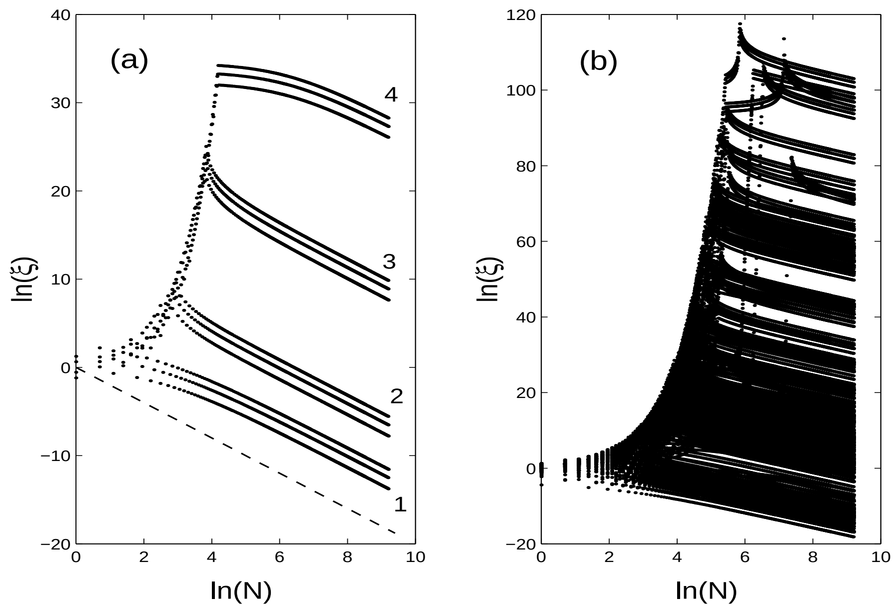

Fig. 1. $\ln(\xi(N))$ vs $\ln(N)$ for $\mu_t = 7/4$. In frame (a) the label 1 (2, 3, 4) refers the initial condition x_0=0 (x_0=0.788149612215968, x_0=0.937389375675543, x_0=0.691124729036456); the dashed line $\xi = N^{-2}$ is a guide for the eyes. In frame (b) the same function is plotted for 100 randomly-chosen initial conditions.

The initial convergence to the attractor strongly depends on the initial condition x_0; for instance frame (a) of Fig. 1 shows that $\xi(N)$ decreases in the first few steps for $x_0 = 0$, while it grows exponentially and for a longer time for other initial conditions. In general most of the initial conditions lead to an initial exponential growth, see frame (b) of Fig. 1. This result, typical of chaos, is not trivial and indicates a not-smooth (probably fractal) basin of convergence to the attractor; as a counterexample there is no initial exponential growth for $\mu = 3/4$.
The same conclusion about the initial regime is reached with a completely different numerical experiment: following Refs. [9,10], we have partitioned the interval $[-1, 1]$ into $W = 10^4$ equal cells, taken $N_0 = 10^6$ random initial points x_0 inside one of the cells (*concentrated initial condition*), and let them evolve according to Eq. (3); then we have repeated the experiment 1000 times randomly changing the initial cell. The initial spread was so fast that more than 5000 cells ($W/2$) were occupied after 30 time steps in 72% of the cases, in agreement with an exponential growth in the initial part of the evolution.

In Refs. [9,10], introducing the generalized Kolmogorov-Sinai entropy $K_q = \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty}(S_q(t) - S_q(0))/t$ and identifying the probability $p_i(t)$ by the accumulation number of the cell-i: $p_i(t) \equiv N_i(t)/N_0$, it was conjec-

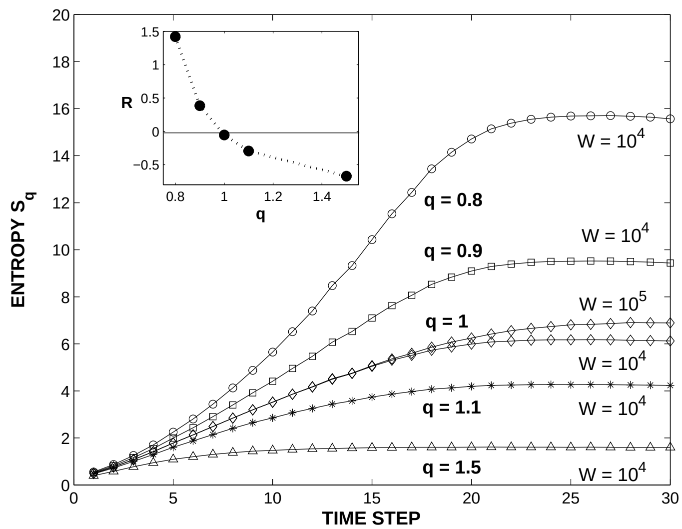

Fig. 2. Entropy evolution for several q's at $\mu = 7/4$ and N_0 initial conditions concentrated in one cell; S_q has been averaged over 1000 initial cells. All curves with $W = 10^4$ have $N_0 = 10^5$, the one curve with $W = 10^5$ ($q = 1$) has $N_0 = 3 \times 10^5$. The inset shows the non-linearity coefficient R vs q for $3 = t_1 < t < t_2 = 12$.

tured [9,10] that K_q is finite in a given system only for a specific value of q, with $q = 1$ corresponding to the chaotic behavior.
We have extended this conjecture in two directions: (a) we have considered the asymptotic behavior of $S_q(t)$ for weakly insensitive points at the edge of chaos, where entropy decreases; (b) we studied $S_q(t)$ in the pre-asymptotic (exponential) convergence to the attractor.

In Fig. 2 we show the pre-asymptotic convergence to the attractor of $S_q(t)$ averaged over the above numerical experiments, all of which had $S_q(0) = 0$ (concentrated initial conditions), for different values of q: only $q = 1$ is compatible with a linear growth of S_q. In fact, the inset shows that the nonlinearity coefficient $R \simeq 0$ for $q = 1$, where $R \equiv C(t_1+t_2)/B$, with $A+Bt+Ct^2$ used to fit $S_q(t)$ between t_1 and t_2. The saturation maximum value would correspond to an uniform distribution $p_i = 1/W$, *e.g.*, $S_{max} = \ln(W)$ for $q = 1$, and increases with W. Both the linear growth of $S_1(t)$ in Fig. 2 and the exponential growth of $\xi(t)$ in Fig. 1 are compatible with a chaotic dynamics of the system in the first pre-asymptotic regime.

Since the asymptotic regime is instead characterized by a power behavior, as shown in Fig. 1, we conjectured that $S_q(t)$ were linear and, therefore, K_q finite only for $q \neq 1$ in these weakly insensitive points, analogously to what happens at the weakly sensitive points.

As proposed in [11], we selected the initial cells that give a faster spread of the distribution: the asymptotic behavior starts from about the maximum value of the entropy and these cells give the dominant contributions. Fig. 3 shows S_q

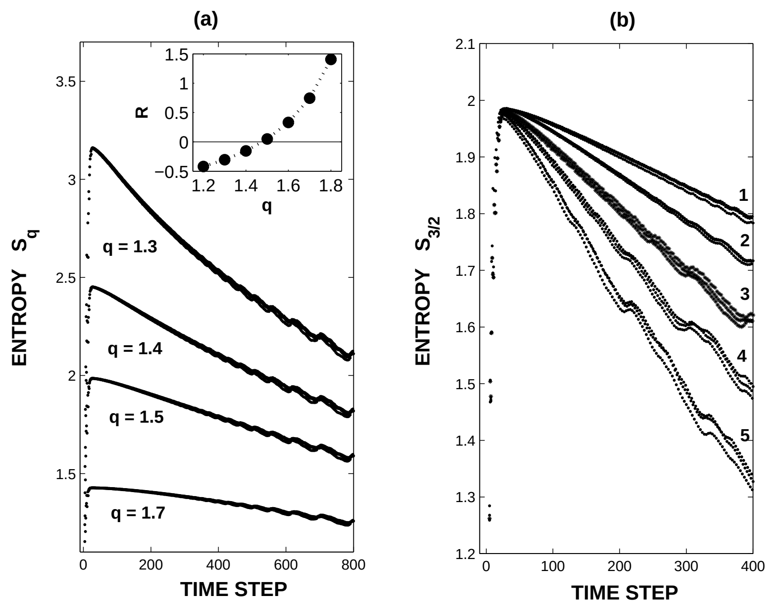

Fig. 3. Entropy evolution at $\mu = 7/4$ with initial conditions concentrated in one cell: (a) for W=512000 and $q = 1.3$, 1.4, 1.5 and 1.7; (b) for $q = 3/2$ and $W = 512000$ (256000, 128000, 64000, 32000) for the ensamble of points 1 (2, 3, 4, 5). Curves 2-5 in (b) have been averaged on five choices of the initial cell, curve 1 has not been averaged. The inset in (a) shows the non-linearity coefficient R vs q for $51 = t_1 < t < t_2 = 550$.

for different values of q and W. S_{max} is approximately reached in the first $\sim$ 30 steps, then S_q decreases. This implies an overshooting in the S_q time evolution. In particular, frame (a) demonstrates that only when $q = q_{sen} = 3/2$ S_q decreases linearly (and K_q is finite): precisely the value expected [1,5] from the attractor structure at $\mu_t = 7/4$ and characterizing the long-time behavior of $\xi(t)$ in Fig. 1. Confirming our hypothesis, the conjecture on the generalized Kolmogorov-Sinai entropy retains its validity also for weakly sensitive conditions at the edge of chaos: a finite $K_q < 0$ is found only for $q = q_{sen} > 1$. Frame (b) in fig. 3 shows that the slope of the linear part of $S_{3/2}$ depends on the number of cells W: the larger W, the longer the linear part of $S_q(t)$ and the more time is needed to reach the asymptotic value.

3 Conclusions

We have studied the sensitivity to initial condition $\xi(t)$ and the entropy at a weakly insensitive point at the edge of chaos ($\mu = 7/4$) finding two regimes.

During the initial pre-asymptotic regime the dynamics mimics a chaotic behavior: the sensitivity to the initial conditions is exponential, the Shannon entropy grows linearly with time, and the Kolmogorov-Sinai entropy is finite

as shown by Figs. 1 and 2.

In the subsequent asymptotic regime $\xi(t)$ decreases with a power law characterized by the index $q = q_{sen} = 3/2$, see Eq. (1) and Fig. 1. The generalized non-extensive q-entropy of Eq. (2) has a linear (decreasing) behavior only for $q = q_{sen} = 3/2$, and for the same q the generalized Kolmogorov-Sinai entropy K_q is finite.

These results suggest that the conjecture on the behavior of entropy and on the value of the entropic index q for weakly sensitive points can be extended also to weakly insensitive points.

We are completing a study of the same problem with the so-called relaxation based approach [8], which appears to corroborate our conclusions.

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